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THE LIMITING VOLUME OF SINGLE INPUT/SINGLE OUTPUT INTERMEDIATE STORAGE IN BATCH PROCESSES UNDER PERIODIC PRODUCTION FAILURE

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Abstract—An analysis has been performed of the capacity of intermediate storage vessels required to buffer the effects of periodic production failure. Simple analytical expressions for the limiting volume of the storage as a function of failure frequency and system parameters have been developed for SISO storage system under the assumption that system variables were integer number. All these simple analytical expressions are directly useful for determining the storage size and are the bases for more advanced engineering study such as; operations research, controller design and process synthesis.

Key words: *Storage, Size, Batch, Periodic, Failure*

INTRODUCTION

Noncontinuous processes have played and will continue to play an important role in the chemical industry because of the flexibility they provide to accommodate significant variability in feed materials, their suitability for producing large number of moderate value products with similar recipes, and their turn-down feature which allows ready adaptation to the inherent seasonability of the market demand for some products.

This kind of process which is intentionally operated in a nonsteady state mode is subject to various process imbalances. In multistage noncontinuous processes without intermediate storage, uninterrupted operation is possible only if the successive stages of processing are perfectly synchronized or the batch equipment itself is used as storage vessels. In this case, installation of intermediate storage will decouple the periodic operation of adjacent batch or semicontinuous units; consequently, intermediate storage can take on an important role for improving operating efficiency. In addition, batch operations are usually subject to higher processing variability and more subject to operator vagaries and error. These kinds of process parameter variations as well as those caused by equipment failure and associated repair times can also be mitigated through intermediate storage if adequate size of intermediate storage and the initial hold-up are chosen appropriately. Moreover, intermediate storage can isolate intermediates when noncontinuous processes are used to produce multiple products in sequential campaigns. Due to these kinds of various roles of intermediate storage, the addition of it gives positive consequences on process operation. However, because of the numerous roles that intermediate storage serves, it is difficult to specify its most effective location and required size in the process.

Simulation techniques are the most common tools employed for this kind of analysis. Numerical simulation using Monte Carlo techniques can be employed for the analysis of intermediate storage availability (Ross 1973). Analytical models which have been studied recently are effective tools for intermediate storage analysis. Karimi and Reklaitis (1983) developed analytical results for the limiting storage volume in serial systems composed of arbitrary configurations of batch, semicontinuous or continuous operations. They extended their results to multiple input/multiple output intermediate storage structure (Karimi and Reklaitis 1985 a) as well as parametric variation case (Karimi and Reklaitis 1985 b, c) The main idea of their successful results was to assume periodic material flow which enabled to use powerful Fourier series properties. The same method has been applied for the periodic material flow including periodic production failure by Lee and Reklaitis (1988 and 1989).

This article represents systematical procedure to obtain the analytic solution of the limiting volume of single input/single output(SISO) intermediate storage under periodic production failures which include no failure case as a subsystem. The unique feature of this study is that the whole modelling step is remarkably simplified by deleting Fourier series development and generalizing production failure pattern. Our main technique to get the goal can be summarized as the Algebra of Modulus Operators and Integer Division Result (APPENDIX A). This study will directly contribute to reducing the overinvestment of storage space and facilities in real chemical industries as well as providing an effective modeling technique to develop inventory control policy in noncontinuous processes.

PROCESS MODELLING

The schematic diagram of relevant process and design variables are shown in Fig. 1. The subscript $i=1$ represents up-stream unit and $i=2$ does down-stream unit. Each unit is supposed to produce a batch of product during every cycle time (ω_i) and after γ_i cycles, production failure of duration (d_i) will occur periodically. The cycle time of a batch unit is composed of a transport time ($x_i\omega_i$) and a non-transport time. The non-transport time is the sum of processing, filling (discharging) and preparation time for up-stream (down-stream) unit and their details are not of interest

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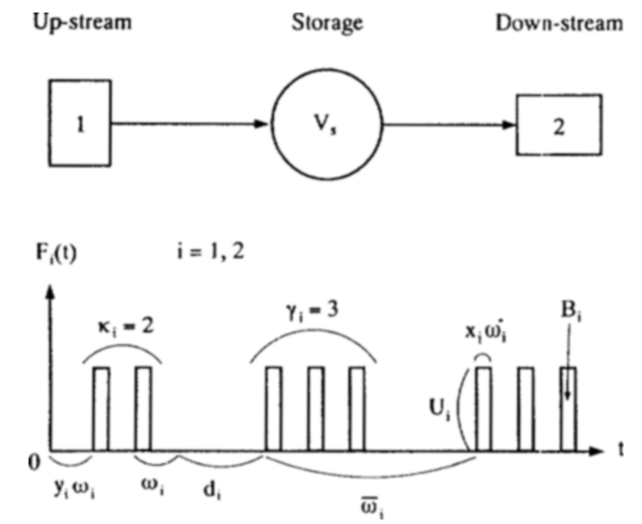

Fig. 1. Modelling of intermediate storage with periodic process failure.

for the purpose of this study. The starting time of inflow from the up-stream unit is assumed to be zero, without loss of generality, and that of the outflow to the down-stream unit is assumed to be $y_2\omega_2$, which is called the initial delay time. The first production failure is assumed to occur after κ_i cycles. It is convenient to define some additional parameters; the overall cycle parameters ($\bar{\omega}_i$) and $\bar{\kappa}_i$, as;

$$\bar{\omega}_i = \gamma_i \omega_i + d_i \tag{1}$$

$$\bar{\kappa}_i = \kappa_i \omega_i + d_i \tag{2}$$

The design parameters for no failure case were based on the use of rational number which is adequate for engineering purpose (Yi 1992). However in this article we have to restrict some of the design parameters to integer value. We will investigate the relaxation of this restriction through simulation study subsequently.

Basic Assumptions;

(i) ω_i and d_i are integers. $i=1, 2$ (3)

(ii) $\mathrm{GCD}(\bar{\omega}_1, \bar{\omega}_2)=1$ (4)

where GCD(.,.) is the greatest common divisor.

The overall material balance on this system gives;

$$\frac{B_1\gamma_1}{\bar{\omega}_1} = \frac{B_2\gamma_2}{\bar{\omega}_2} \tag{5}$$

The flow pattern, which include periodic production failure, is shown in Fig. 1. The material balance around the storage unit reduces to a simple ordinary differential equation (Yi 1992).

$$\frac{dV(t)}{dt} = F_1(t)u[t-\bar{\kappa}_1] + \kappa_1 B_1 - F_2(t)u[t-\bar{\kappa}_2 - y_2\omega_2] - \kappa_2 B_2 \tag{6}$$

where $F_i(t)$ is defined in Fig. 1 and u[.] is unit step function.

There are three situations to be considered in integrating Eq. (6) as shown in Fig. 2. The present time t can occur either during a failure time, during a non-transport time or during a transport time period. We can count the number of complete batches and calculate the incomplete batch size in each case via modulation operators. Combining three expressions with the minimum function produces the following integration of flow under periodic failures;

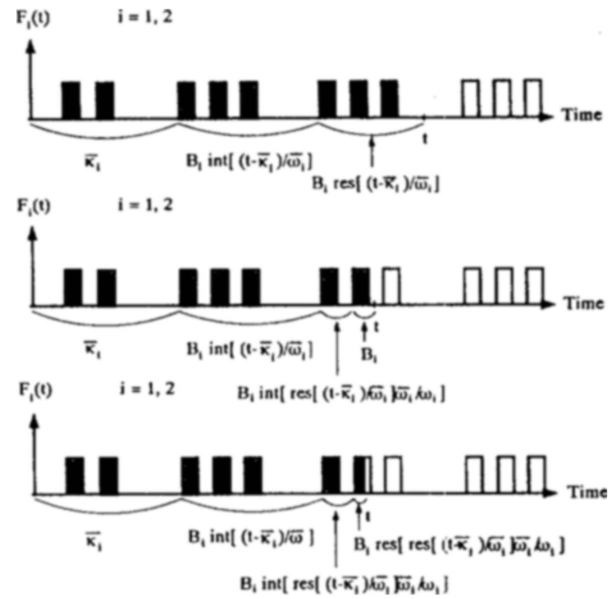

Fig. 2. Integration of material flow function with periodic process failure.

$$\int_0^{t-\bar{\kappa}_i-y_i\omega_i} F_i(t)dt = B_i\Big(\gamma_i \,\mathrm{int}\Big[\frac{t-\bar{\kappa}_i-y_i\omega_i}{\bar{\omega}_i}\Big] + \min\Big\{\gamma_i, \mathrm{int}\Big[\mathrm{res}\Big[\frac{t-\bar{\kappa}_i-y_i\omega_i}{\bar{\omega}_i}\Big]\frac{\bar{\omega}_i}{\omega_i}\Big] + \min\Big\{1, \frac{1}{x_i}\mathrm{res}\Big[\mathrm{res}\Big[\frac{t-\bar{\kappa}_i-y_i\omega_i}{\bar{\omega}_i}\Big]\frac{\bar{\omega}_i}{\omega_i}\Big]\Big\}\Big\}\Big) \tag{7}$$

The hold-up equation for this system is;

$$V(t) = V(0) + \int_0^{t-\bar{\kappa}_1} F_1(t)dt - \int_0^{t-\bar{\kappa}_2-y_2\omega_2} F_2(t)dt + \kappa_1 B_1 - \kappa_2 B_2 \tag{8}$$

where $t \geq \max\{\bar{\kappa}_1, \bar{\kappa}_2 + y_2\omega_2\}$.

The maximum and minimum hold-up is necessary in order to calculate storage size. This hold-up equation is periodic with the period of $\overline{\omega_1\omega_2}$. The local optimal points must occur at the edge points of flow within one period, the same as no failure case (Karimi and Reklaitis 1983). Thus, it can be shown that:

If $U_1 \geq U_2$ (Up-stream Dominant Case)

$$t^1_{min} + \alpha_1\bar{\omega}_1 + \delta_1\omega_1 + \bar{\kappa}_1 \tag{9}$$

$$t^1_{max} = \alpha_1\bar{\omega} + \delta_1\omega_1 + \bar{\kappa}_1 + x_1\omega_1 \tag{10}$$

where $0 \leq \alpha_1 \leq \bar{\omega}_2 - 1$, $0 \leq \delta_1 \leq \gamma_1 - 1$

If $U_1 < U_2$ (Down-stream Dominant Case)

$$t^2_{min} = \alpha_2\bar{\omega}_2 + \delta_2\omega_2 + y_2\omega_2 + \bar{\kappa}_2 + x_2\omega_2 \tag{11}$$

$$t^2_{max} = \alpha_2\bar{\omega}_2 + \delta_2\omega_2 + y_2\omega_2 + \bar{\kappa}_2 \tag{12}$$

where $0 \leq \alpha_2 \leq \bar{\omega}_1 - 1$, $0 \leq \delta_2 \leq \gamma_2 - 1$

The continuous search variable t can be changed into the finite integer search variables α_i and δ_i by inserting Eqs. (9)-(12) into Eq. (7) and (8).

There are four cases which must be considered in order to carry out algebraic manipulation: minimum hold-up in up-stream dominant case, minimum hold-up in down-stream dominant case, maximum hold-up in up-stream dominant case and maximum hold-up in down-stream dominant case. There is great similarity between the four cases but they do differ in detail. The first case will be considered here and the others are given APPENDIX B.

MINIMUM HOLD-UP IN UP-STREAM DOMINANT CASE ($U_1 \geq U_2$)

Eq. (9) is inserted into Eq. (7) and each term in Eq. (7) can be developed further such as;

$$\frac{t^1_{mn}-\bar{\kappa}_1}{\bar{\omega}_1}=\alpha_1+\frac{\delta_1\omega_1}{\bar{\omega}_1} \tag{13}$$

$$\mathrm{int}\left(\frac{t^1_{mn}-\bar{\kappa}_1}{\bar{\omega}_1}\right)=\alpha_1$$

$$\mathrm{res}\left(\frac{t^1_{mn}-\bar{\kappa}_1}{\bar{\omega}_1}\right)=\frac{\delta_1\omega_1}{\bar{\omega}_1}$$

$$\mathrm{int}\left[\mathrm{res}\left(\frac{t^1_{mn}-\bar{\kappa}_1}{\bar{\omega}_1}\right)\frac{\bar{\omega}_1}{\omega_1}\right]=\delta_1$$

$$\mathrm{res}\left[\mathrm{res}\left(\frac{t^1_{mn}-\bar{\kappa}_1}{\bar{\omega}_1}\right)\frac{\bar{\omega}_1}{\omega_1}\right]=0 \tag{14}$$

$$\frac{t^1_{mn}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}=\frac{\alpha_1\bar{\omega}_1+\delta_1\omega_1+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1}{\bar{\omega}_2}+\frac{1-\mathrm{res}[y_2\omega_2]}{\bar{\omega}_2} \tag{15}$$

Two steps of modulation procedure for α_1 have to be applied to convert search variables α_1 and δ_1 into more convenient search variables, α_1' and α_1'', following the procedure of modulation of variables given in APPENDIX A.

$$\alpha_1\bar{\omega}_1+\delta_1\omega_1+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1=q_1'\bar{\omega}_2+\alpha_1' \tag{16}$$

where $0\leq\alpha_1'\leq\bar{\omega}_2-1$.

$$\alpha_1'=q_1''\omega_2+\alpha_1'' \tag{17}$$

where $0\leq\alpha_1''\leq\omega_2-1$.

The down-stream terms in Eq. (7) can be resolved via new search variables.

$$\mathrm{int}\left[\frac{t^1_{mn}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]=q_1'$$

$$\mathrm{res}\left[\frac{t^1_{mn}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]=\frac{\alpha_2'+1-\mathrm{res}[y_2\omega_2]}{\bar{\omega}_2}$$

$$\mathrm{int}\left[\mathrm{res}\left[\frac{t^1_{mn}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]\frac{\bar{\omega}_2}{\omega_2}\right]=q_1''$$

$$\mathrm{res}\left[\mathrm{res}\left[\frac{t^1_{mn}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]\frac{\bar{\omega}_2}{\omega_2}\right]=\frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{\omega_2} \tag{18}$$

Inserting Eq. (7), (14) and (18) into (8) produces;

$$\begin{aligned}V(t^1_{mn})=&\left(B_1\gamma_1-\frac{B_2\gamma_2\bar{\omega}_1}{\bar{\omega}_2}\right)\alpha_1+\left(B_1-\frac{B_2\gamma_2\omega_1}{\bar{\omega}_2}\right)\delta_1\\&+V(0)+\kappa_1B_1-\kappa_2B_2-\frac{B_2\gamma_2(\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2}\\&-B_2\left(-\frac{\gamma_2\alpha_1'}{\bar{\omega}_2}+\min\left\{\gamma_2,\ q_1''\right.\right.\\&\left.\left.+\min\left\{1,\ \frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2}\right\}\right\}\right)\end{aligned} \tag{19}$$

The first term of Eq. (19) can be removed using Eq. (6). The coefficient of δ_1 in the second term of Eq. (19) is positive which means the minimum of the hold-up function occurs when δ_1 is minimum:

$$\begin{aligned}\min_{\delta_1}V(t^1_{mn})(\delta_1)=&V(t^1_{mn})(\delta_1=0)\\=&V(0)+\kappa_1B_1-\kappa_2B_2-\frac{B_2\gamma_2(\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2}\\&+B_2\left[-\frac{\gamma_2\alpha_1'}{\bar{\omega}_2}+\min\left\{\gamma_2,\ q_1''+\min\left\{1,\right.\right.\right.\\&\left.\left.\left.\frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2}\right\}\right\}\right]\end{aligned} \tag{20}$$

It is necessary to separate the span of our independent search variables into the three parts within which the hold-up is linear with respect to the search variables in Eq. (20), namely; $\{\alpha_1'\geq\gamma_2\omega_2\}$, $\{\alpha_1'<\gamma_2\omega_2$ and $x_2\omega_2-1+\mathrm{res}[y_2\omega_2]\leq\alpha_1''\}$ and $\{\alpha_1'<\gamma_2\omega_2$ and $x_2\omega_2-1+\mathrm{res}[y_2\omega_2]\geq\alpha_1''\}$.

(i) $\alpha_1'\geq\gamma_2\omega_2$ (21)

In this case, the following useful relationships are valid:

$$q_1''\geq\gamma_2 \tag{22a}$$

$$\min\{\gamma_2,\ q_1''+\cdots\}=\gamma_2 \tag{22b}$$

Inserting Eq. (22b) into (20) produces;

$$\begin{aligned}V(t^1_{mn})=&V(0)+\kappa_1B_1-\kappa_2B_2-\frac{B_2\omega_2(\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2}\\&+B_2\left(\frac{\gamma_2\alpha_1'}{\bar{\omega}_2}-\gamma_2\right)\end{aligned} \tag{23}$$

Now, α_1' is the only remaining search variable. The minimum of hold-up occurs when α_1' is minimum because the coefficient of α_1' is positive.

$$\begin{aligned}\min_{\alpha_1'}V(t^1_{mn})(\alpha_1')=&V(t^1_{mn})(\alpha_1'=\gamma_2\omega_2)=V^1_{mn}\\=&V(0)+\kappa_1B_1-(\kappa_2+\gamma_2)B_2\\&-\frac{B_2\gamma_2(-\gamma_2\omega_2+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2}\end{aligned} \tag{24}$$

(ii) $\alpha_1'<\gamma_2\omega_2$ and $x_2\omega_2-1+\mathrm{res}[y_2\omega_2]\leq\alpha_1''$ (25)

In this case, the following useful relationships are valid:

$$\min\left\{1,\ \frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2}\right\}=1 \tag{26a}$$

$$\min\{\gamma_2,\ q_1''+\min\{\cdots\}\}=q_1''+1 \tag{26b}$$

Inserting Eq. (26b) and (17) into (20) produces;

$$\begin{aligned}V(t^1_{mn})=&V(0)+\kappa_1B_1-\kappa_2B_2-\frac{B_2\gamma_2(\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2}\\&+B_2\left[\frac{\gamma_2\alpha_1''}{\bar{\omega}_2}+\left(\frac{\gamma_2\omega_2}{\bar{\omega}_2}-1\right)q_1''-1\right]\end{aligned} \tag{27}$$

Since the coefficient of α_1'' is positive and that of q_1'' is negative, α_1'' has to be decreased and q_1'' increased to minimize hold-up. Thus,

$$\min\ \alpha_1''=\mathrm{int}[(x_2+y_2)\omega_2]-\mathrm{int}[y_2\omega_2] \tag{28a}$$

$$\max\ q_1''=\gamma_2-1 \tag{28b}$$

Then, the minimum hold-up is:

$$\min_{\alpha_1'', q_1''} V(t^1_{min})(\alpha_1'', q_1'')$$
$$= V(t^1_{min})(\min\ \alpha_1'', \max\ q_1'') = V^2_{min}$$
$$= V(0)+\kappa_1 B_1-(\kappa_2+\gamma_2)B_2 - \frac{B_2\gamma_2(-(\gamma_2-1)\omega_2+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[(x_2+y_2)\omega_2]-1)}{\bar{\omega}_2} \quad (29)$$

The left side of second inequality in Eq. (25) should not be greater than ω_2-1. A restriction is imposed on Eq. (29).

RESTRICTION S1: $\mathrm{res}[y_2\omega_2]\le(1-x_2)\omega_2$

(iii) $\alpha_1'<\gamma_2\omega_2$ and $x_2\omega_2-1+\mathrm{res}[y_2\omega_2]\ge\alpha_1''$ (30)

In this case, the following useful relationships are valid:

$$\min\left\{1,\ \frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2}\right\}=\frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2} \quad (31a)$$

$$\min\{\gamma_2,\ q_1''+\min\{\cdots\}\}=q_1''+\frac{\alpha_1''+1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2} \quad (31b)$$

Inserting Eq. (31b) and (17) into (20) produces;

$$V(t^1_{min})=V(0)+\kappa_1B_1-\kappa_2B_2-\frac{B_2\gamma_2(\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2}$$
$$+B_2\left[\left(\frac{\gamma_2}{\bar{\omega}_2}-\frac{1}{x_2\omega_2}\right)\alpha_1''+\left(\frac{\gamma_2\omega_2}{\bar{\omega}_2}-1\right)q_1''-\frac{1-\mathrm{res}[y_2\omega_2]}{x_2\omega_2}\right] \quad (32)$$

Since the coefficients of α_1'' and q_1'' are negative, α_1'' and q_1'' have to be increased to minimize hold-up. Thus,

$$\max\ \alpha_1''=\mathrm{int}[(x_2+y_2)\omega_2]-\mathrm{int}[y_2\omega_2]-1 \quad (33a)$$

$$\max\ q_1''=\gamma_2-1 \quad (33b)$$

Then, the minimum hold-up is;

$$\max_{\alpha_1', q_1'} V(t^1_{min})(\alpha_1'', q_1'')$$
$$=V(t^1_{min})(\max\ \alpha_1'', \max\ q_1'')=V^3_{min}$$
$$=V(0)+\kappa_1B_1-(\kappa_2+\gamma_2)B_2+\frac{B_2\,\mathrm{res}[(x_2+y_2)\omega_2]}{x_2\omega_2}-\frac{B_2\gamma_2(-(\gamma_2-1)\omega_2+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_2} \quad (34)$$

Since the left side of second inequality in Eq. (30) should not be less than 0, a restriction is imposed on Eq. (34).

RESTRICTION S2: $x_2\omega_2\ge1-\mathrm{res}[y_2\omega_2]$

Eqs. (24), (29) and (34) give three candidates for the global minimum hold-up subject to two parametric restrictions. Careful comparison between these equations reveals that the hold-up calculated by Eq. (29) is less than that by Eq. (24). If system parameters violate RESTRICTION S1, Eq. (29) is not valid and Eq. (24) and (34) are the candidate of global minimum. If system parameters violate RESTRICTION S2, Eq. (34) is not valid and Eq. (29) is the only global minimum.

STORAGE VOLUME

Summarizingly, all pertinent equations from the previous section and APPENDIX B, we obtain:

Up-stream Dominant Case ($U_1\ge U_2$)

$$V^1_{max}=V(0)+(\kappa_1+\gamma_1)B_1-(\kappa_2+\gamma_2)B_2-\frac{B_2\gamma_2(-(\bar{\omega}_2-1)+(\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2} \quad (A32)$$

$$V^2_{max}=V(0)+(\kappa_1+\gamma_1)B_1-(\kappa_2+1)B_2-\frac{B_2\gamma_2(-(\omega_2-1)+(\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2} \quad (A36)$$

$$V^3_{max}=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2-\frac{B_2\,\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}-\frac{B_2\gamma_2((\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2} \quad (A40)$$

$$V^1_{min}=V(0)+\kappa_1B_1-(\kappa_2+\gamma_2)B_2-\frac{B_2\gamma_2(-\gamma_2\omega_2+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[y_2\omega_2]-1)}{\bar{\omega}_2} \quad (24)$$

$$V^2_{min}=V(0)+\kappa_1B_1-(\kappa_2+\gamma_2)B_2-\frac{B_2\gamma_2(-(\gamma_2-1)\omega_2+\bar{\kappa}_1-\bar{\kappa}_2-\mathrm{int}[(x_2+y_2)\omega_2]-1)}{\bar{\omega}_2} \quad (29)$$

$$V^3_{min}=V(0)+\kappa_1B_1-(\kappa_2+\gamma_2)B_2+\frac{B_2\,\mathrm{res}[(x_2+y_2)\omega_2]}{x_2\omega_2}-\frac{B_2\gamma_2(-(\gamma_2-1)\omega_2+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_2} \quad (34)$$

Down-stream Dominant Case ($U_1\le U_2$)

$$V^1_{min}=V(0)+(\kappa_1+\gamma_1)B_1-(\kappa_2+\gamma_2)B_2+\frac{B_1\gamma_1(-(\bar{\omega}_1-1)+(\gamma_2-1)\omega_2-\bar{\kappa}_1+\bar{\kappa}_2+\mathrm{int}[x_2+y_2)\omega_2])}{\bar{\omega}_1} \quad (A12)$$

$$V^2_{min}=V(0)+(\kappa_1+1)B_1-(\kappa_2+\gamma_2)B_2-\frac{B_2\gamma_2(-(\omega_1-1)+(\gamma_2-1)\omega_2-\bar{\kappa}_1+\bar{\kappa}_2+\mathrm{int}[x_2+y_2)\omega_2])}{\bar{\omega}_2} \quad (A16)$$

$$V^3_{min}=V(0)+\kappa_1B_1-(\kappa_2+\gamma_2)B_2+\frac{B_1\,\mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1}+\frac{B_1\gamma_1((\gamma_2-1)\omega_2-\bar{\kappa}_1+\bar{\kappa}_2+\mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \quad (A20)$$

$$V^1_{max}=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2+\frac{B_1\gamma_1(-\gamma_1\omega_1-\bar{\kappa}_1+\bar{\kappa}_2+\mathrm{int}[y_2\omega_2])}{\bar{\omega}_1} \quad (A52)$$

$$V^2_{max}=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2+\frac{B_1\gamma_1(-(\gamma_1-1)\omega_1-\bar{\kappa}_1+\bar{\kappa}_2-\mathrm{int}[x_1\omega_1-y_2\omega_2]-1)}{\bar{\omega}_1} \quad (A57)$$

$$V^3_{max}=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2-\frac{B_1\,\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_1\omega_1}+\frac{B_1\gamma_1(-(\gamma_1-1)\omega_1-\bar{\kappa}_1+\bar{\kappa}_2-\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_1} \quad (A62)$$

Careful comparison of three maximum/minimum hold-up expressions shows that one of them can be deleted and the remaining two expressions can be combined into one to yield:

Up-stream Dominant Case ($U_1\ge U_2$)

$$V^2_{max}\le V^1_{max} \quad (35a)$$

$$V^1_{min} \geq V^2_{min} \tag{35b}$$

$$\begin{aligned} V_{max} &= \max\{V^1_{max}, V^3_{max}\} \\ &= V(0) + (\kappa_1+\gamma_1)B_1 - \kappa_2 B_2 \\ &\quad - \frac{B_2\gamma_2}{\bar{\omega}_2}\{(\gamma_1-1)\omega_1 + \bar{\kappa}_1 - \bar{\kappa}_2 + \mathrm{int}[x_1\omega_1 - y_2\omega_2]\} \\ &\quad - \frac{B_2\gamma_2}{\bar{\omega}_2}\min\left\{1, \frac{\bar{\omega}_2 \mathrm{res}[x_1\omega_1 - y_2\omega_2]}{x_2\gamma_2\omega_2}\right\} \end{aligned} \tag{36}$$

$$\begin{aligned} V_{min} &= \min\{V^2_{min}, V^3_{min}\} \\ &= V(0) + \kappa_1 B_1 - (\kappa_2+\gamma_2)B_2 \\ &\quad - \frac{B_2\gamma_2}{\bar{\omega}_2}\{-(\gamma_2-1)\omega_2 + \bar{\kappa}_1 - \bar{\kappa}_2 + \mathrm{int}[(x_2+y_2)\omega_2]\} \\ &\quad + \frac{B_2\gamma_2}{\bar{\omega}_2}\min\left\{1, \frac{\bar{\omega}_2 \mathrm{res}[(x_2+y_2)\omega_2]}{x_2\gamma_2\omega_2}\right\} \end{aligned} \tag{37}$$

Dowm-stream Dominant Case ($U_1 \leq U_2$)

$$V^1_{max} \leq V^2_{max} \tag{38a}$$

$$V^2_{min} \geq V^1_{min} \tag{38b}$$

$$\begin{aligned} V_{max} &= \max\{V^2_{max}, V^3_{max}\} \\ &= V(0) + (\kappa_1+\gamma_1)B_1 - \kappa_2 B_2 \\ &\quad - \frac{B_2\gamma_2}{\bar{\omega}_2}\{(\gamma_1-1)\omega_1 + \bar{\kappa}_1 - \bar{\kappa}_2 + \mathrm{int}[x_1\omega_1 - y_2\omega_2]\} \\ &\quad - \frac{B_2\gamma_2}{\bar{\omega}_2}\min\left\{1, \frac{\bar{\omega}_1 \mathrm{res}[x_1\omega_1 - y_2\omega_2]}{x_1\gamma_1\omega_1}\right\} \end{aligned} \tag{39}$$

$$\begin{aligned} V_{min} &= \min\{V^1_{min}, V^3_{min}\} \\ &= V(0) + \kappa_1 B_1 - (\kappa_2+\gamma_2)B_2 \\ &\quad - \frac{B_2\gamma_2}{\bar{\omega}_2}\{-(\gamma_1-1)\omega_2 + \bar{\kappa}_1 - \bar{\kappa}_2 + \mathrm{int}[(x_2+y_2)\omega_2]\} \\ &\quad + \frac{B_2\gamma_2}{\bar{\omega}_2}\min\left\{1, \frac{\bar{\omega}_1 \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\gamma_1\omega_1}\right\} \end{aligned} \tag{40}$$

Comparing the equations between two cases shows that they are the same except for the terms $\left(\frac{\bar{\omega}_1}{x_1\gamma_1\omega_1}\right)$ or $\left(\frac{\bar{\omega}_2}{x_2\gamma_2\omega_2}\right)$. If we set the parameters to those of the no failure case, namely $\gamma_i = 1$, $\kappa_i = \bar{\kappa}_i = 0$, $\omega_1 = \bar{\omega}_1 = \beta_2$ and $\omega_2 = \bar{\omega}_2 = \beta_1$, then, the equations reduce to exactly the same as those of no failure case which were developed independently in Yi (1992).

$$V_{max} = V(0) + B_1 - \frac{B_1}{\beta_2}\left(\mathrm{int}[x_1\beta_2 - y_2\beta_1] + \min\left\{1, \frac{\mathrm{res}[x_1\beta_2 - y_2\beta_1]}{x_k}\right\}\right) \tag{41}$$

$$V_{min} = V(0) + B_2 - \frac{B_1}{\beta_2}\left(\mathrm{int}[(x_2+y_2)\beta_1] + \min\left\{1, \frac{\mathrm{res}[(x_2+y_2)\beta_1]}{x_k}\right\}\right) \tag{42}$$

where $k=1$ for $U_1 \leq U_2$
$k=2$ for $U_1 > U_2$

It should be noted that Eq. (41) and (42) are still valid when the cycle times are rational number and are not prime to each other in spite of the basic assumptions (i) and (ii).

Let us emphasize the restrictions and their related equations again.

RESTRICTION S1:	$\mathrm{res}[y_2\omega_2] \leq (1-x_2)\omega_2$	Eq. (29)
RESTRICTION S2:	$x_2\omega_2 \geq 1 - \mathrm{res}[y_2\omega_2]$	Eq. (34)
RESTRICTION S3:	$(1-x_1) > 1 - \mathrm{res}[(x_2+y_2)\omega_2]$	Eq. (A16)
RESTRICTION S4:	$x_1\omega_1 \geq \mathrm{res}[(x_2+y_2)\omega_2]$	Eq. (A20)
RESTRICTION S5:	$1 - \mathrm{res}[x_1\omega_1 - y_2\omega_2] \leq (1-x_2)\omega_2$	Eq. (A36)
RESTRICTION S6:	$x_2\omega_2 \geq \mathrm{res}[x_1\omega_1 - y_2\omega_2]$	Eq. (A40)
RESTRICTION S7:	$(1-x_1)\omega_1 \geq 1 - \mathrm{res}[y_2\omega_2]$	Eq. (A57)
RESTRICTION S8:	$x_1\omega_1 \geq \mathrm{res}[y_2\omega_2]$	Eq. (A62)

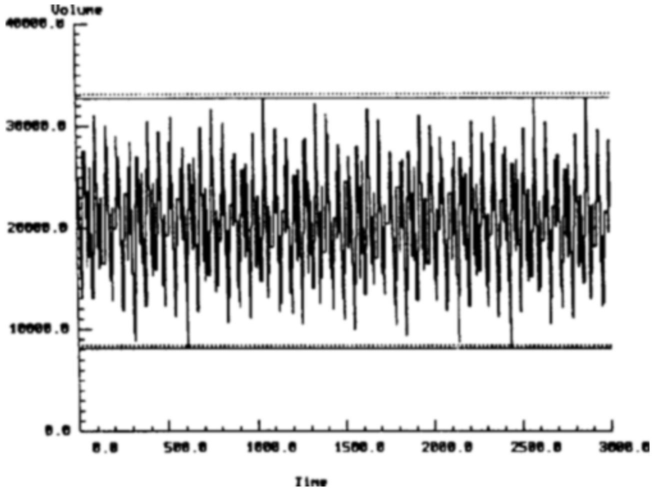

Fig. 3. Hold-up profile of example with real valued cycle times.

Each restriction is connected with the validity of the equation from which it derives. For example, if design parameters violate RESTRICTION S1, Eq. (29) is not available and the minimum hold-up can be calculated by Eq. (24) and (34) in up-stream dominant case. RESTRICTION S3 and S5 are not necessary because their corresponding equations can not become the maximum or minimum of hold-up from Eqs. (35a) and (38b). The global maximum or minimum when system parameters violate one of the restrictions can be summarized as follows:

Up-stream Dominant Case ($U_1 \geq U_2$)

Violation of RESTRICTION S1..	$V_{min} = \min\{V^3_{min}, V^1_{min}\}$	(43a)
Violation of RESTRICTION S2..	$V_{min} = V^2_{min}$	(43b)
Violation of RESTRICTION S6..	$V_{max} = V^1_{max}$	(43c)

Down-stream Dominant Case ($U_1 \leq U_2$)

Violation of RESTRICTION S4..	$V_{min} = V^1_{min}$	(44a)
Violation of RESTRICTION S7..	$V_{max} = \max\{V^3_{max}, V^1_{min}\}$	(44b)
Violation of RESTRICTION S8..	$V_{max} = V^2_{max}$	(44c)

If the restrictions are examined carefully, they eliminate the parametric domain with large x_2 (RESTRICTION S1), small x_2 (RESTRICTION S2 and S6), large x_1 (RESTRICTION S7) and small x_1 (RESTRICTION S4 and S8) which are unusual in practice.

The maximum/minimum hold-up equations developed in this study can be utilized to calculate the storage size and initial inventory in conjunction with the following conditions (Karimi and Reklaitis 1983):

$$V_{max} \leq V, \tag{45a}$$

$$V_{min} \geq 0 \tag{45b}$$

DISCUSSION AND SIMULATION EXAMPLES

The basic assumption (i) [Eq. (3)] restricts cycle times and fail-

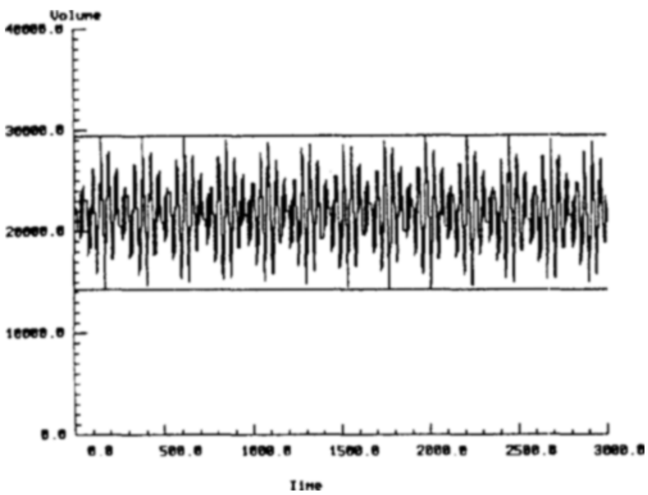

Fig. 4. Hold-up profile of example with integer valued cycle times.

ure durations to integer values in the SISO storage under periodic failure. The storage designer can scale the system parameters to make cycle times and failure times integer valued but this can be very inconvenient. Extensive simulation has shown that this integer restriction can not be relaxed to rational numbers in general. Specifically, the simulation showed that if the basic assumptions are violent, the maximum/minimum hold-up calculated using rational numbers would approximate the actual values within 10 % error. Fig. 3 shows one example with rational numbers for the cycle time and failure duration. The solid lines represent the actual maximum or minimum hold-up and the dashed lines represent the maximum or minimum hold-up calculated using the equations developed in this study.

Fig. 4 shows the hold-up profile of an example which includes integer valued periodic failure and integer valued cycle times. The solid lines represent the exact maximum and minimum hold-up calculated using Eqs. (36), (37), (39), (40), (43) and (44).

The basic assumption (ii) that overall cycle times are prime with respect to each other is also inconvenient. The simulation showed that the maximum/minimum hold-up calculated by the equations developed in this study was the upper/lower bound of the exact value when the overall cycle times were not prime with respect to each other.

CONCLUSIONS

The analytical storage sizing equations have been developed for SISO storage under periodical production failure. The cycle times and periodic failure times are assumed to be integer value in order to carry out logical development. The resulting equations with no failure parameter setting were completely matched with the equations of no failure case which were developed independently. Therefore, intermediate storage system with periodic production failure was proved to include that of no failure case as a subsystem.

The results of this study is going to be the basis of designing controller or operating algorithm of failure prone intermediate storage as well as storage sizing.

APPENDIX A: Modulation of Variables

1. Algebra of Modulus Operators

$A = int[A] + res[A]$ where $0 \le res[A] < 1$

$$int[A \pm int[B]] = int[A] \pm int[B]$$
$$res[A \pm int[B]] = res[A]$$

If $res[A] \ge res[B]$

$$int[A-B] = int[A] - int[B]$$
$$res[A-B] = res[A] - res[B]$$

If $res[A] < res[B]$

$$int[A-B] = int[A] - int[B] - 1$$
$$res[A-B] = res[A] - res[B] + 1$$

If $res[A] + res[B] < 1$

$$int[A+B] = int[A] + int[B]$$
$$res[A+B] = res[A] + res[B]$$

If $res[A] + res[B] \ge 1$

$$int[A+B] = int[A] + int[B] + 1$$
$$res[A+B] = res[A] + res[B] - 1$$

2. Integer Division Result

For the given real z and integers β_1, β_2, the integer variable α_1 maps one to one and onto the integer variables (q_1, r_1) such that;

$$\alpha_1\beta_1 + int[z] = q_1\beta_2 + r_1$$

where

$$\alpha_1 = \{0,\ 1,\ 2,\ \cdots,\ \beta_2 - 1\}$$
$$r_1 = \{0,\ 1,\ 2,\ \cdots,\ \beta_2 - 1\}$$

if and only if $GCD(\beta_1, \beta_2) = 1$.

For proof, see Burton (1970).

The following working expression can be developed using the above result.

$$int\left[\frac{\alpha_1\beta_1 + z}{\beta_2}\right] = q_1 = \frac{\alpha_1\beta_1 + int[z] - r_1}{\beta_2}$$

$$res\left[\frac{\alpha_1\beta_1 + z}{\beta_2}\right] = \frac{r_1 + res[z]}{\beta_2}$$

APPENDIX B: Sizing Equations for SISO Storage

1. Minimum Hold-up in Down-stream Dominant Case ($U_1 \le U_2$)

$$\frac{t^2_{min} - \bar{\kappa}_1}{\bar{\omega}_1} = \frac{\alpha_2\bar{\omega}_2 + \delta_2\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + int[(x_2+y_2)\omega_2]}{\bar{\omega}_1} + \frac{res[(x_2+y_2)\omega_2]}{\bar{\omega}_1} \quad (A1)$$

Two steps of modulation have to be applied for α_2.

$$\alpha_2\bar{\omega}_2 + \delta_2\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + int[(x_2+y_2)\omega_2] = q_2'\bar{\omega}_1 + \alpha_2' \quad (A2)$$

where $0 \le \alpha_2' \le \bar{\omega}_1 - 1$

$$\alpha_2' = q_2''\omega_1 + \alpha_2'' \quad (A3)$$

where $0 \le \alpha_2'' \le \omega_1 - 1$

$$\mathrm{int}\left[\frac{t_{min}^2 - \bar{\kappa}_1}{\bar{\omega}_1}\right] = q_2'$$

$$\mathrm{res}\left[\frac{t_{min}^2 - \bar{\kappa}_1}{\bar{\omega}_1}\right] = \frac{\alpha_2' + \mathrm{res}[(x_2+y_2)\omega_2]}{\bar{\omega}_1}$$

$$\mathrm{int}\left[\mathrm{res}\left[\frac{t_{min}^2 - \bar{\kappa}_1}{\bar{\omega}_1}\right]\frac{\bar{\omega}_1}{\omega_1}\right] = q_2''$$

$$\mathrm{res}\left[\mathrm{res}\left[\frac{t_{min}^2 - \bar{\kappa}_1}{\bar{\omega}_1}\right]\frac{\bar{\omega}_1}{\omega_1}\right] = \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{\omega_1} \tag{A4}$$

$$\frac{t_{min}^2 - \bar{\kappa}_2 - y_2\omega_2}{\bar{\omega}_2} = \alpha_2 + \frac{(\delta_2 + x_2)\omega_2}{\bar{\omega}_2} \tag{A5}$$

$$\mathrm{int}\left[\frac{t_{min}^2 - \bar{\kappa}_2 - y_2\omega_2}{\bar{\omega}_2}\right] = \alpha_2$$

$$\mathrm{res}\left[\frac{t_{min}^2 - \bar{\kappa}_2 - y_2\omega_2}{\bar{\omega}_2}\right] = \frac{(\delta_2 + x_2)\omega_2}{\bar{\omega}_2}$$

$$\mathrm{int}\left[\mathrm{res}\left[\frac{t_{min}^2 - \bar{\kappa}_2 - y_2\omega_2}{\bar{\omega}_2}\right]\frac{\bar{\omega}_2}{\omega_2}\right] = \delta_2$$

$$\mathrm{res}\left[\mathrm{res}\left[\frac{t_{min}^2 - \bar{\kappa}_2 - y_2\omega_2}{\bar{\omega}_2}\right]\frac{\bar{\omega}_2}{\omega_2}\right] = x_2 \tag{A6}$$

Inserting Eq. (7) in the main text, (A4) and (A6) into (8) in the main text gives;

$$\begin{aligned} V(t_{min}^2) = &\left(\frac{B_1\gamma_1\bar{\omega}_2}{\bar{\omega}_1} - B_2\gamma_2\right)\alpha_2 + \left(\frac{B_1\gamma_1\omega_2}{\bar{\omega}_1} - B_2\right)\delta_2 \\ &+ V(0) + \kappa_1 B_1 - \kappa_2 B_2 - B_2 \\ &+ \frac{B_1\gamma_1(\bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \\ &+ B_1\left(-\frac{\gamma_1\alpha_2'}{\bar{\omega}_1} + \min\left\{\gamma_1,\ q_2'' \right.\right. \\ &\left.\left. + \min\left\{1,\ \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1}\right\}\right\}\right) \end{aligned} \tag{A7}$$

The first term of Eq. (A7) can be removed by Eq. (6) in the main text and the coefficient of δ_2 in Eq. (A7) is negative which means;

$$\begin{aligned} \min_{\delta_2} V(t_{min}^2)(\delta_2) &= V(t_{min}^2)(\max\ \delta_2) = V(t_{min}^2)(\delta_2 = \gamma_2 - 1) \\ &= V(0) + \kappa_1 B_1 - \kappa_2 B_2 - B_2 + \left(\frac{B_1\gamma_1\omega_2}{\bar{\omega}_1} - B_2\right)(\gamma_2 - 1) \\ &+ \frac{B_1\gamma_1(\bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \\ &- B_1\left(\frac{\gamma_1\alpha_2'}{\bar{\omega}_1} - \min\left\{\gamma_1,\ q_2'' \right.\right. \\ &\left.\left. + \min\left\{1,\ \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1}\right\}\right\}\right) \end{aligned} \tag{A8}$$

We have to separate the span of searching variables into three parts to develop minimum functions.

(i) $\alpha_2' \ge \gamma_1\omega_1$ (A9)

$$q_2'' \ge \gamma_1 \tag{A10a}$$

$$\min\{\gamma_1,\ q_2'' + \cdots\} = \gamma_1 \tag{A10b}$$

Inserting Eq. (A10) into (A8) produces;

$$\begin{aligned} V(t_{min}^2) = &V(0) + \kappa_1 B_1 - (\kappa_2 + \gamma_2)B_2 \\ &+ \frac{B_1\omega_1((\gamma_2 - 1)\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \\ &- B_1\left(\frac{\gamma_1\alpha_2'}{\bar{\omega}_1} - \gamma_1\right) \end{aligned} \tag{A11}$$

The coefficient of α_2' is negative which means;

$$\begin{aligned} \min_{\alpha_2'} V(t_{min}^2)(\alpha_2') &= V(t_{min}^2)(\max\ \alpha_2' = \bar{\omega}_1 - 1) = V_{min}^1 \\ &= V(0) + (\kappa_1 + \gamma_1)B_1 - (\kappa_2 + \gamma_2)B_2 \\ &- \frac{B_1\gamma_1((1 - \bar{\omega}_1 + (\gamma_2 - 1)\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \end{aligned} \tag{A12}$$

(ii) $\alpha_2' < \gamma_1\omega_1$ and $x_1\omega_1 - \mathrm{res}[(x_2+y_2)\omega_2] \le \alpha_2''$ (A13)

$$\min\left\{1,\ \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1}\right\} = 1 \tag{A14a}$$

$$\min\{\gamma_1,\ q_2'' + \min\{\cdots\}\} = q_2'' + 1 \tag{A14b}$$

Inserting Eq. (A14) into (A8) produces;

$$\begin{aligned} V(t_{min}^2) = &V(0) + \kappa_1 B_1 - (\kappa_2 + \gamma_2)B_2 \\ &+ \frac{B_1\gamma_1((\gamma_2 - 1)\omega_2 + \bar{\kappa}_1 - \bar{\kappa}_2 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \\ &+ B_1\left[-\frac{\gamma_1\alpha_2''}{\bar{\omega}_1} - \left(\frac{\gamma_1\omega_1}{\bar{\omega}_1} - 1\right)q_2'' + 1\right] \end{aligned} \tag{A15}$$

The coefficient of α_2'' is negative and that of q_2'' is positive which means;

$$\begin{aligned} \min_{\alpha_2'', q_2''} V(t_{min}^2)(\alpha_2'',\ q_2'') &= V(t_{min}^2)(\max\ \alpha_2'' = \omega_1 - 1,\ \min\ q_2'' = 0) = V_{min}^2 \\ &= V(0) + (\kappa_1 + 1)B_1 - (\kappa_2 + \gamma_2)B_2 \\ &+ \frac{B_1\gamma_1(1 - \omega_1 + (\gamma_2 - 1)\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \end{aligned} \tag{A16}$$

The left side of second inequality in Eq. (A13) should not be greater than $\omega_1 - 1$. This imposes some restriction for Eq. (A 16).

RESTRICTION S3: $1 - \mathrm{res}[(x_2+y_2)\omega_2] \le (1 - x_1)\omega_1$

(iii) $\alpha_2' < \gamma_1\omega_1$ and $x_1\omega_1 - \mathrm{res}[(x_2+y_2)\omega_2] \ge \alpha_2''$ (A17)

$$\min\left\{1,\ \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1}\right\} = \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1} \tag{A18a}$$

$$\min\{\gamma_1,\ q_2'' + \min\{\cdots\}\} = q_2'' + \frac{\alpha_2'' + \mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1} \tag{A18b}$$

Inserting Eq. (A18) into (A8) produces;

$$\begin{aligned} V(t_{min}^1) = &V(0) + \kappa_1 B_1 - (\kappa_2 + \gamma_2)B_2 + \frac{B_1\,\mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1} \\ &+ \frac{B_1\gamma_1((\gamma_2 - 1)\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \\ &+ B_1\left[\left(\frac{1}{x_1\omega_1} - \frac{\gamma_1}{\bar{\omega}_1}\right)\alpha_2'' + \left(1 - \frac{\gamma_1\omega_1}{\bar{\omega}_1}\right)q_2''\right] \end{aligned} \tag{A19}$$

The coefficients of α_2'' and q_2'' are positive which means;

$$\begin{aligned} \min_{\alpha_2'', q_2''} V(t_{min}^2)(\alpha_2'',\ q_2'') &= V(t_{min}^2)(\min\ \alpha_2'' = 0,\ \min\ q_2'' = 0) = V_{min}^3 \\ &= V(0) - \kappa_1 B_1 - (\kappa_2 + \gamma_2)B_2 + \frac{B_1\,\mathrm{res}[(x_2+y_2)\omega_2]}{x_1\omega_1} \\ &+ \frac{B_1\gamma_1((\gamma_2 - 1)\omega_2 + \bar{\kappa}_2 - \bar{\kappa}_1 + \mathrm{int}[(x_2+y_2)\omega_2])}{\bar{\omega}_1} \end{aligned} \tag{A20}$$

The left side of second inequality in Eq. (A17) should not be less than 0. This imposes some restriction for Eq. (A20).

RESTRICTION S4: $x_1\omega_1 \geq \mathrm{res}[(x_2+y_2)\omega_2]$

2. Maximum Hold-up in Up-stream Dominant Case ($U_1 \geq U_2$)

$$\frac{t^1_{max}-\bar{\kappa}_1}{\bar{\omega}_1}=\alpha_1+\frac{(\delta_1+x_1)\omega_1}{\bar{\omega}_1} \qquad \text{(A21)}$$

$$\mathrm{int}\left[\frac{t^1_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]=\alpha_1$$

$$\mathrm{res}\left[\frac{t^1_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]=\frac{(\delta_1+x_1)\omega_1}{\bar{\omega}_1}$$

$$\mathrm{int}\left[\mathrm{res}\left[\frac{t^1_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]\frac{\bar{\omega}_1}{\omega_1}\right]=\delta_1$$

$$\mathrm{res}\left[\mathrm{res}\left[\frac{t^1_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]\frac{\bar{\omega}_1}{\omega_1}\right]=x_1 \qquad \text{(A22)}$$

$$\frac{t^1_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}=\frac{\alpha_1\bar{\omega}_1+\delta_1\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2]}{\bar{\omega}_1}+\frac{\mathrm{res}[x_1\omega_1-y_2\omega_2]}{\bar{\omega}_2} \qquad \text{(A23)}$$

We have to apply two steps of modulation procedure for α_1.

$$\alpha_1\bar{\omega}_1+\delta_1\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2]=q_1'\bar{\omega}_2+\alpha_1' \qquad \text{(A24)}$$

where $0 \leq \alpha_1' \leq \bar{\omega}_2-1$

$$\alpha_1'=q_1''\omega_2+\alpha_1'' \qquad \text{(A25)}$$

where $0 \leq \alpha_1'' \leq \omega_2-1$

$$\mathrm{int}\left[\frac{t^1_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]=q_1'$$

$$\mathrm{res}\left[\frac{t^1_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_1}\right]=\frac{\alpha_1'+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{\bar{\omega}_2}$$

$$\mathrm{int}\left[\mathrm{res}\left[\frac{t^1_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]\frac{\bar{\omega}_2}{\omega_2}\right]=q_1''$$

$$\mathrm{res}\left[\mathrm{res}\left[\frac{t^1_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_1}\right]\frac{\bar{\omega}_2}{\omega_2}\right]=\frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{\omega_2} \qquad \text{(A26)}$$

Inserting Eq. (7) in the main text, (A22) and (A26) into (8) in the main text produces;

$$\begin{aligned}V(t^1_{max})=&\left(B_1\gamma_1-\frac{B_2\gamma_2\bar{\omega}_1}{\bar{\omega}_2}\right)\alpha_1+\left(B_1-\frac{B_2\gamma_2\omega_1}{\bar{\omega}_2}\right)\delta_1+B_1\\&+V(0)+\kappa_1B_1-\kappa_2B_2-\frac{B_2\gamma_2(\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\\&-B_2\left(-\frac{\gamma_2\alpha_1'}{\bar{\omega}_2}+\min\left\{\gamma_2,\ q_1''+\min\left\{1,\ \frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}\right\}\right\}\right)\end{aligned} \qquad \text{(A27)}$$

The first term of Eq. (A27) can be removed by Eq. (6) in the main text and the coefficient of δ_1 in the second term of Eq. (A27) is positive which means;

$$\begin{aligned}\max_{\delta_1}V(t^1_{max})(\delta_1)&=V(t^1_{max})(\max\delta_1=\gamma_1-1)\\&=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2\\&\quad-\frac{B_2\gamma_2((\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\\&\quad+B_2\left(-\frac{\gamma_2\alpha_1'}{\bar{\omega}_2}-\min\left\{\gamma_2,\ q_1''+\min\left\{1,\ \frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}\right\}\right\}\right)\end{aligned} \qquad \text{(A28)}$$

We have to separate the span of our independent searching variables into three parts to evaluate the minimum functions in Eq. (A28).

(i) $\alpha_1' \geq \gamma_2\omega_2$ (A29)

$$q_1'' \geq \gamma_2 \qquad \text{(A30a)}$$

$$\min\{\gamma_2,\ q_1''+\cdots\}=\gamma_2 \qquad \text{(A30b)}$$

Inserting Eq. (A30) into (A28) produces;

$$\begin{aligned}V(t^1_{max})=&V(0)+(k_1+\gamma_1)B_1-k_2B_2\\&-\frac{B_2\gamma_2((\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\\&+B_2\left(\frac{\gamma_2\alpha_1'}{\bar{\omega}_2}-\gamma_2\right)\end{aligned} \qquad \text{(A31)}$$

$$\begin{aligned}&\max_{\alpha_1'}V(t^1_{max})(\alpha_1')\\&=V(t^1_{max})(\max\ \alpha_1'=\bar{\omega}_2-1)=V^1_{max}\\&=V(0)+(\kappa_1+\gamma_1)B_1-(\kappa_2+\gamma_2)B_2\\&\quad-\frac{B_2\gamma_2(1-\bar{\omega}_2+(\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\end{aligned} \qquad \text{(A32)}$$

(ii) $\alpha_1'<\gamma_2\omega_2$ and $x_2\omega_2-\mathrm{res}[x_1\omega_1-y_2\omega_2] \leq \alpha_1''$ (A33)

$$\min\left\{1,\ \frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}\right\}=1 \qquad \text{(A34a)}$$

$$\min\{\gamma_2,\ q_1''+\min\{\cdots\}\}=q_1''+1 \qquad \text{(A34b)}$$

Inserting Eq. (A34) into (A28) produces;

$$\begin{aligned}V(t^1_{max})=&V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2\\&-\frac{B_2\gamma_2((\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\\&+B_2\left[\frac{\gamma_2\alpha_1''}{\bar{\omega}_2}+\left(\frac{\gamma_2\omega_2}{\bar{\omega}_2}-1\right)q_1''-1\right]\end{aligned} \qquad \text{(A35)}$$

The coefficient of α_1'' is positive and that of q_1'' is negative. This means;

$$\begin{aligned}&\min_{\alpha_1''q_1''}V(t^1_{max})(\alpha_1'',\ q_1'')\\&=V(t^1_{max})(\max\ \alpha_1''=\omega_2-1,\ \min\ q''=0)=V^2_{max}\\&=V(0)+(\kappa_1+\gamma_1)B_1-(\kappa_2+1)B_2)\\&\quad-\frac{B_2\gamma_2((1-\omega_2+(\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\end{aligned} \qquad \text{(A36)}$$

The left side of second inequality in Eq. (A33) should not be greater than ω_2-1. This imposes some restriction for Eq. (A36).

RESTRICTION S5: $(1-x_2)\omega_2 \geq 1-\mathrm{res}[x_1\omega_1-y_2\omega_2]$

(iii) $\alpha_1'<\gamma_2\omega_2$ and $x_2\omega_2-\mathrm{res}[x_1\omega_1-y_2\omega_2] \geq \alpha_1''$ (A37)

$$\min\left\{1,\ \frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}\right\}=\frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2} \qquad \text{(A38a)}$$

$$\min\{\gamma_2,\ q_1''+\min\{\cdots\}\}=q_1''+\frac{\alpha_1''+\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_2\omega_2} \qquad \text{(A38b)}$$

Inserting Eq. (A38) into (A28) produces;

$$\begin{aligned}V(t^1_{max})=&V(0)+(\kappa_1+\gamma_1)B_2-\kappa_2B_2\\&-\frac{B_2\gamma_2((\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2}\end{aligned}$$

$$+B_2\left[\left(\frac{\gamma_2}{\bar{\omega}_2}-\frac{1}{x_2\omega_2}\right)\alpha_1''+\left(\frac{\gamma_2\omega_2}{\bar{\omega}_2}-1\right)q_1''-\frac{res[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}\right] \tag{A39}$$

The coefficients of α_1'' and q_1'' are negative. This means α_1'' and q_1'' have to be decreased to maximize V.

$$\max_{\alpha_1'',q_1''} V(t^1_{max})(\alpha_1'', q_1'')=V(t^1_{max})(\min \alpha_1''=0,\ \min q_1''=0)=V^3_{max}$$
$$=V(0)+(\kappa_1+\gamma_1)B_1-K_2B_2-\frac{B_2\,res[x_1\omega_1-y_2\omega_2]}{x_2\omega_2}-\frac{B_2\gamma_2((\gamma_1-1)\omega_1+\bar{\kappa}_1-\bar{\kappa}_2+int[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_2} \tag{A40}$$

The left side of second inequality in Eq. (A37) should not be less than 0. This imposes some restriction for Eq. (A40).

RESTRICTION S6: $x_2\omega_2 \geq res[x_1\omega_1-y_2\omega_2]$

3. Maximum Hold-up in Down-stream Dominant Case ($U_1 \leq U_2$)

$$\frac{t^2_{max}-\bar{\kappa}_1}{\bar{\omega}_1}=\frac{\alpha_2\bar{\omega}_2+\delta_2\omega_2+\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2]+res[y_2\omega_2]}{\bar{\omega}_1} \tag{A41}$$

Two steps of modulation have to be applied for α_2.

$$\alpha_2\bar{\omega}_2+\delta_2\omega_2+\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2]=q_2'\bar{\omega}_1+\alpha_2' \tag{A42}$$

where $0\leq\alpha_2'\leq\bar{\omega}_1-1$

$$\alpha_2'=q_2''\omega_1+\alpha_2'' \tag{A43}$$

where $0\leq\alpha_2''\leq\omega_1-1$

$$int\left[\frac{t^2_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]=q_2'$$
$$res\left[\frac{t^2_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]=\frac{\alpha_2'+res[y_2\omega_2]}{\bar{\omega}_1}$$
$$int\left[res\left[\frac{t^2_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]\frac{\bar{\omega}_1}{\omega_1}\right]=q_2''$$
$$res\left[res\left[\frac{t^2_{max}-\bar{\kappa}_1}{\bar{\omega}_1}\right]\frac{\bar{\omega}_1}{\omega_1}\right]=\frac{\alpha_2''+res[y_2\omega_2]}{\omega_1} \tag{A44}$$

$$\frac{t^2_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}=\alpha_2+\frac{\delta_2\omega_2}{\bar{\omega}_2} \tag{A45}$$

$$int\left[\frac{t^2_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]=\alpha_2$$
$$res\left[\frac{t^2_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_2}\right]=\frac{\delta_2\omega_2}{\bar{\omega}_2}$$
$$int\left[res\left[\frac{t^2_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_1}\right]\frac{\bar{\omega}_2}{\omega_2}\right]=\delta_2$$
$$res\left[res\left[\frac{t^2_{max}-\bar{\kappa}_2-y_2\omega_2}{\bar{\omega}_1}\right]\frac{\bar{\omega}_2}{\omega_2}\right]=0 \tag{A46}$$

Inserting Eq. (7) in the main text, (A44) and (A46) into (8) in the main text gives;

$$V(t^2_{max})=\left(\frac{B_1\gamma_1\bar{\omega}_2}{\bar{\omega}_1}-B_2\gamma_2\right)\alpha_2+\left(\frac{B_1\gamma_1\omega_2}{\bar{\omega}_1}-B_2\right)\delta_2+V(0)+\kappa_1B_1-\kappa_2B_2+\frac{B_1\gamma_1(\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2])}{\bar{\omega}_1}$$
$$+B_1\left(-\frac{\gamma_1\alpha_2'}{\bar{\omega}_1}+\min\left\{\gamma_1,\ q_2''+\min\left\{1,\ \frac{\alpha_2''+res[y_2\omega_2]}{x_1\omega_1}\right\}\right\}\right) \tag{A47}$$

The first term of Eq. (A47) can be removed by Eq. (6) in the main text and the coefficient of δ_2 in Eq. (A47) is negative which means;

$$\min_{\delta_2}V(t^1_{max})(\delta_2)=V(t^2_{max})(\min\ \delta_2=0)$$
$$=V(0)+\kappa_1B_1-\kappa_2B_2+\frac{B_1\gamma_1(\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2])}{\bar{\omega}_1}+B_1\left(-\frac{\gamma_1\alpha_2'}{\bar{\omega}_1}+\min\left\{\gamma_1,\ q_2''+\min\left\{1,\ \frac{\alpha_2''+res[y_2\omega_2]}{x_1\omega_1}\right\}\right\}\right) \tag{A48}$$

We have to separate the span of searching variables into three parts to develop minimum functions.

(i) $\alpha_2'\geq\gamma_1\omega_1$ (A49)

$$q_2''\geq\gamma_1 \tag{A50a}$$
$$\min\{\gamma_1,\ q_2''+\cdots\}=\gamma_1 \tag{A50b}$$

Inserting Eq. (A50) into (A48) produces;

$$V(t^2_{max})=V(0)+\kappa_1B_1-\kappa_2B_2+\frac{B_1\omega_1(\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2])}{\bar{\omega}_1}-B_1\left(\frac{\gamma_1\alpha_2'}{\bar{\omega}_1}-\gamma_1\right) \tag{A51}$$

The coefficient of α_1' is negative which means;

$$\max_{\alpha_2}V(t^2_{max})(\alpha_2')=V(t^2_{max})(\min\ \alpha_2'=\gamma_1\omega_1)=V^1_{max}$$
$$=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2+\frac{B_1\gamma_1(-\gamma_1\omega_1+\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2])}{\bar{\omega}_1} \tag{A52}$$

(ii) $\alpha_2'<\gamma_1\omega_1$ and $x_1\omega_1-res[y_2\omega_2]\leq\alpha_2''$ (A53)

$$\min\left\{1,\ \frac{\alpha_2''+res[y_2\omega_2]}{x_1\omega_1}\right\}=1 \tag{A54a}$$
$$\min\{\gamma_1,\ q_2''+\min\{\cdots\}\}=q_2''+1 \tag{A54b}$$

Inserting Eq. (A54) into (A48) produces;

$$V(t^2_{max})=V(0)+\kappa_1B_1-\kappa_2B_2+\frac{B_1\gamma_1(\bar{\kappa}_2-\bar{\kappa}_1+int[y_2\omega_2])}{\bar{\omega}_1}+B_1\left[-\frac{\gamma_1\alpha_2''}{\bar{\omega}_1}+\left(1-\frac{\gamma_1\omega_1}{\bar{\omega}_1}\right)q_2''+1\right] \tag{A55}$$

The coefficient of α_2'' is negative and that of q_2'' is positive. Then, maximum hold-up occurs minimum of α_2'' and maximum of q_2''.

$$\min\ \alpha_2''=int[x_1\omega_1-y_2\omega_2]+int[y_2\omega_2]+1 \tag{A56a}$$
$$\max\ q_2''=\gamma_1-1 \tag{A56b}$$

$$\max_{\alpha_2'',q_2''}V(t^2_{max})(\alpha_2'',\ q_2'')$$
$$=V(t^2_{max})(\min\ \alpha_2'',\ \max q_2'')=V^2_{max}$$
$$=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2+\frac{B_1\gamma_1(-(\gamma_1-1)\omega_1+\bar{\kappa}_2-\bar{\kappa}_1+int[x_1\omega_1-y_2\omega_2]-1)}{\bar{\omega}_1} \tag{A57}$$

The left side of second inequality in Eq. (A53) should not be greater than ω_1-1. This imposes some restriction for Eq. (A57).

RESTRICTION S7: $1-\mathrm{res}[y_2\omega_2]\leq(1-x_1)\omega_1$

(iii) $\alpha_2'<\gamma_1\omega_1$ and $x_1\omega_1-\mathrm{res}[y_2\omega_2]\geq\alpha_2''$ (A58)

$$\min\left\{1,\ \frac{\alpha_2''+\mathrm{res}[y_2\omega_2]}{x_1\omega_1}\right\}=\frac{\alpha_2''+\mathrm{res}[y_2\omega_2]}{x_1\omega_1}$$

$$\min\{\gamma_1,\ q_2''+\min\{\cdots\}\}=q_2''+\frac{\alpha_2''+\mathrm{res}[y_2\omega_2]}{x_1\omega_1} \quad \text{(A59)}$$

Inserting Eq. (93) into (82) produces;

$$V(t^2_{max})=V(0)+\kappa_1B_1-\kappa_2B_2+\frac{B_1\mathrm{res}[y_2\omega_2]}{x_1\omega_1}+\frac{B_1\gamma_1(\bar{\kappa}_2-\bar{\kappa}_1+\mathrm{int}[y_2\omega_2])}{\bar{\omega}_1}+B_1\left[\left(\frac{1}{x_1\omega_1}-\frac{\gamma_1}{\bar{\omega}_1}\right)\alpha_2''+\left(1-\frac{\gamma_1\omega_1}{\bar{\omega}_1}\right)q_2''\right] \quad \text{(A60)}$$

The coefficients of α_2'' and q_2'' are positive. The maximum values of α_2'' and q_2'' are required to maximize V.

$$\max\ \alpha_2''=\mathrm{int}[x_1\omega_1-y_2\omega_2]+\mathrm{int}[y_2\omega_2] \quad \text{(A61a)}$$

$$\max\ q_2''=\gamma_1-1 \quad \text{(A61b)}$$

$$\max_{\alpha_2'',\,q_2''} V(t^2_{max})(\alpha_2'',\ q_2'')=V(t^2_{max})(\min\ \alpha_2'',\ \max q_2'')=V^3_{max}$$
$$=V(0)+(\kappa_1+\gamma_1)B_1-\kappa_2B_2-\frac{B_1\mathrm{res}[x_1\omega_1-y_2\omega_2]}{x_1\omega_1}+\frac{B_1\gamma_1(-(\gamma_1-1)\omega_1+\bar{\kappa}_2-\bar{\kappa}_1-\mathrm{int}[x_1\omega_1-y_2\omega_2])}{\bar{\omega}_1} \quad \text{(A62)}$$

The left side of second inequality in Eq. (A58) should not be less than 0. This imposes some restriction for Eq. (A62).

RESTRICTION S8: $x_1\omega_1\geq\mathrm{res}[y_2\omega_2]$

NOMENCLATURE

B_i : batch size
d_i : process failure time
$F_i(t)$: material flow function
q_i : integer quotient of the search variable
U_i : flow rate
V_{max} : maximum hold-up
V_{min} : minimum hold-up
V^i_{max} : one among three solutions of maximum hold-up
V(t) : hold-up function
V(0) : initial hold-up
V_s : storage size
x_i : transportation time fraction
y_i : initial delay time fraction
z : arbitrary real number

Greek Letters

α_i : integer search variable
β_i : integer
δ_i : integer search variable
γ_i : number of batches between process failure
κ_i : number of batches before the first failure
$\bar{\kappa}_i$: defined by Eq. (2)
ω_i : cycle time
$\bar{\omega}_i$: defined by Eq. (1)

Subscript

i : 1 for up-streams, 2 for down-streams

Special Functions

GCD(,,) : greatest common divisor
int[.] : truncation function to make integer
res[.] : residual function to be truncated
u[.] : unit step function

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